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# A NOTE

# ON ERDŐS'S CONJECTURE ON MULTIPLICITIES OF COMPLETE SUBGRAPHS LOWER UPPER BOUND FOR CLIQUES OF SIZE 6

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## 1. The purpose of the note

Denote by $k_t(G)$ the number of cliques of order $t$ in graph $G$. Let $k_t(n) = min\{k_t(G)+k_t(\overline{G}) : |G|=n\}$, where $\overline{G}$ denotes the complement of $G$, and $|G|$ denotes the order of $G$. Let $c_t(n)=k_t(n)/\binom{n}{t}$, and let $c_t = \lim_{n\to\infty} c_t(n)$. An old conjecture of Erdős, related to Ramsey's theorem, states that $c_t=2^{1-\binom{t}{2}}$. It was shown false by Thomason for all $t \geq 4$ ([3],[4]). Franek and Rödl ([1]) presented a simpler counterexample to the conjecture for $t=4$ derived from a simple Cayley graph of order $2^{10}$ obtained by a computer search giving essentially the same upper bound for $c_4$ as Thomason's. In this note we show that the same graph gives rise to two sequences of graphs, one a counteraxmple for $t=5$ and the other for $t=6$ improving the original Thomason's $c_5<0.906\cdot 2^{-9}$ to $c_5\leq 0.885834\cdot 2^{-9}$ (though Jagger, Thomason, and Števíček [2] obtained a better $c_5\leq 0.8801\cdot 2^{-9}$), and Thomason's original $c_6<0.936\cdot 2^{-14}$ to $c_6\leq 0.744514\cdot 2^{-14}$ (though meanwhile [2] gave a bit worse $0.7641\cdot 2^{-14}$). If *weak Rödl's conjecure* that $c_t 2^{\binom{t}{2}}\to 0$ is true, then the bounds of the Ramsey number $r(t,t)$ improve, while if the *strong Rödl's conjecture* that $c_t 2^{\binom{t}{2}}\to 0$ *exponentially fast* is true, then the bounds of $r(t,t)$ improve exponentially. The interesting aspects of the new and previous bounds for

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$c_5$ and $c_6$ is that they corroborate Rödl's conjecture. It is interesting to mention that the referee of this note obtained $c_7 \leq 0.715527$ for the same graph, though it had not been verified yet.

## 2. A brief description of the method

The method from [1] was used again. The vertices of graph $G$ are all subsets of $X = \{1,2,3,4,5,6,7,8,9,10\}$. $x, y \subset X$ form an edge if and only if $|x \triangle y| \in F = \{1,3,4,7,8,10\}$, where $\triangle$ denotes the operation of symmetric difference. A sequence $\{nG\}$ of graphs is constructed from $G$ in the same way as described in [4] or [1]. It is not then hard to verify that

$$c_5 \leq \lim_{n\to\infty} \frac{k_5(nG) + k_5(\overline{nG})}{\binom{|nG|}{5}} = \frac{120(k_5(G) + k_5(\overline{G})) + 240k_4(G) + 150k_3(G) + 30k_2(G) + |G|}{|G|^5}$$

and

$$c_6 \leq \lim_{n\to\infty} \frac{k_6(nG) + k_6(\overline{nG})}{\binom{|nG|}{6}} = \frac{720(k_6(G) + k_6(\overline{G})) + 1800k_5(G) + 1560_4(G) + 540k_3(G) + 62k_2(G) + |G|}{|G|^6}.$$

Since we cannot compute $k_t(G)$ directly, we instead computed a number of (ordered) sequences of subsets of $X$, $\langle x_1, \cdots, x_t\rangle$, so that $|x_i| \in F$ and $|x_i \triangle x_j| \in F$. This is based on an observation that $k_{t+1}(G) = \frac{2^{10}}{(t+1)!} s_t(F)$ (and $k_{t+1}(\overline{G}) = \frac{2^{10}}{(t+1)!} s_t(\overline{F})$), where $s_t$ is the number of such sequences of length $t$. The sequences were counted by being generated by a computer program (see [1]). Thus,

$$c_5 \leq \frac{s_4(F) + s_4(\overline{F}) + 10s_3(F) + 25s_2(F) + 15s_1(F) + 1}{2^{40}}$$

and

$$c_6 \leq \frac{s_5(F) + s_5(\overline{F}) + 15s_4(F) + 65s_3(F) + 90s_2(F) + 31s_1(F) + 1}{2^{50}}$$

Since computer-generated results that cannot be easily verified are always suspect, an utmost care was used in checking the programs. First, the routines to calculate $s_t(F)$ can be checked (and were) whether they work properly by using $F = \{1,2,3,4,5,6,7,8,9,10\}$ which should lead to a value of

$(2^{10}-1)(2^{10}-2)\dots(2^{10}-t)$. Second, the values were calculated by a two independently written set of programs in a span of three years. Thus, we can be reasonably confident in the results. The results were obtained using various large SUN machines, the first set of programs was written in C and the other was written in C++. The calculations required use of arbitrary precision numbers, however with the exception of the computations of $s_t(F)$, they all can be done manually.

## 3. Results

Upper bound for $c_5$:
cardinality family: $F=\{1,3,4,7,8,10\}$
$s_1=506$, $s_2=125730$, $s_3=14734170$, $s_4=742203000$
complementary cardinality family: $\overline{F}=\{2,5,6,9\}$
$s_4=1009617840$
numerator = 1902313381
denominator = 2147483648 $(2^{31}=2^{40-9})$
result = 0.8858336978591978549957275390625

Upper bound for $c_6$:
cardinality family: $F=\{1,3,4,7,8,10\}$
$s_1=506$, $s_2=125730$, $s_3=14734170$, $s_4=742203000$, $s_5=13677741000$
complementary cardinality family: $\overline{F}=\{2,5,6,9\}$
$s_5=25382760480$
numerator = 51162598917
denominator = 68719476736 $(2^{36}=2^{50-14})$
result = 0.744513802303117699921131134033203125

## References


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